Homework Assignment 4
Solutions

1. The null hypothesis is $H_0 : \mu = 18\text{mg}$ and the alternative is $H_a : \mu < 18\text{mg}$. The test statistic is
\[ z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{14.68 - 18}{4.2/\sqrt{45}} = -5.30. \]
Since we have a one-sided test, the critical value is $z_{0.01} = -2.33$. Thus, we reject the null hypothesis at the 1% significance level and conclude that adult females under the age of 51 are, on average, getting less than the RDA of 18 mg of iron.
[ 2 points ]

2. Please also see the code.
   (a) The probability that any single lab will not reject the null hypothesis is 0.95. Assuming each lab is independent of the others, the probability that no lab rejects is $0.95^{50} = 0.077$.
   (b) The probability that any single lab will reject the null hypothesis is 0.05. Assuming independence between the labs, the probability that every lab rejects the null is $0.05^{50} = 9 \times 10^{-66}$.
   (c) The probability that 10 labs reject the null can be calculated with the Binomial distribution, with $n=50$ and $p=0.05$, which is 0.00013.
[ 2 points ]

3. Please also see the code.
   (a) Only one hypothesis test would be considered significant when controlling the FWER at 5% using the Bonferroni correction (the one with $p=0.003$).
   (b) When controlling the FDR at 5% with Benjamini-Hochberg, four brain regions would be called significant (the ones with $p < 0.016$).
[ 3 points ]

4. We have
\[
E \left[ \hat{\sigma}^2 \right] = E \left[ \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2 + (n_3 - 1)S_3^2}{n_1 + n_2 + n_3 - 3} \right]
\]
\[
= \frac{1}{n_1 + n_2 + n_3 - 3} \times E \left[ (n_1 - 1)S_1^2 + (n_2 - 1)S_2^2 + (n_3 - 1)S_3^2 \right]
\]
\[
= \frac{1}{n_1 + n_2 + n_3 - 3} \times (n_1 - 1)E \left[ S_1^2 \right] + (n_2 - 1)E \left[ S_2^2 \right] + (n_3 - 1)E \left[ S_3^2 \right]
\]
\[
= \frac{1}{n_1 + n_2 + n_3 - 3} \times (n_1 - 1)\sigma^2 + (n_2 - 1)\sigma^2 + (n_3 - 1)\sigma^2
\]
\[
= \frac{1}{n_1 + n_2 + n_3 - 3} \times (n_1 + n_2 + n_3 - 3)\sigma^2
\]
\[
= \sigma^2
\]
When the expected value of an estimator equals the quantity being estimated, the estimator is unbiased.

[ 3 points ]

5. Please also see the code. The 95% confidence interval for the difference in ferulic acid concentration is \((-45.49; -0.51)\), which does not cover zero. In the (somewhat unusual) scenario when the sample sizes and the sample standard deviations are the same for both groups, it does not matter whether or not we assume that the within-group variances are the same in truth - the degrees of freedom and the standard errors are mathematically the same. See the class notes - in this setting (identical sample sizes and sample standard deviations) the estimated standard errors in both cases simplify to \(S \times \sqrt{2/n}\), and the approximated degrees of freedom simplify to

\[
\frac{(2S^2/n)^2}{2(S^2/n)^2/(n-1)} = (n-1) \times \frac{4S^4/n^2}{2S^4/n^2} = 2 \times (n-1) = 2n - 2.
\]

[ 3 points ]

6. Please see the code. Under the assumption of independence the probability that at least one specific test shows a significant difference at the 5% level is \(1 - 0.95^{77} \approx 0.98\). If none of the alternative hypothesis were true, one would still expect 77 \(\times 0.05 = 3.85\) significant results on average, so 2 out of 77 would not be very surprising. If one or both of the p-values below 0.05 were less than the Bonferroni-cutoff of 0.05/77 = 0.00065, the researchers could still claim a significant finding even if the independence assumption is false.

[ 2 points ]

7. Please see the code. The 95% confidence interval for the treatment effect is \((46.8; 152.8)\). The p-value is 0.005. We conclude that there is a treatment effect.

[ 2 points ]