Topic 1: Multiple Linear Regression

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- Review Simple Linear Regression (SLR) and Multiple Linear Regression (MLR) with two predictors
- More Review of MLR via a detailed example
- Model checking for MLR
 - Keywords: MLR, scatterplot matrix, regression coefficient, 95% confidence interval, t-test, adjustment, adjusted variables plot, residual, dbeta, influence

2. Learning objectives

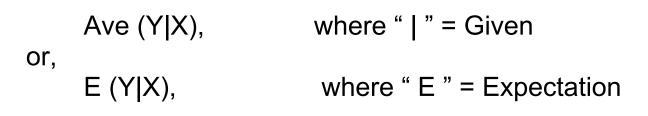
- Understand and explain what MLR is and how it is used to draw inferences
- Interpret MLR coefficients correctly
- Critically evaluate a multiple linear regression analysis to ensure that substantive findings are appropriate given the data
- Interpret the effects of length of stay and employee salary on per capita health care expenditures

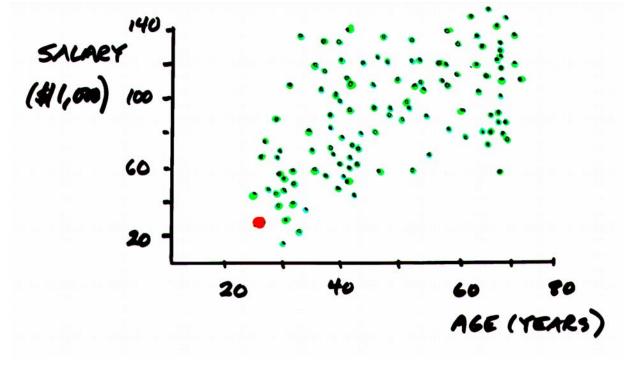
3. What is regression?

• What is the "Regression" of Y on X ?

— Average Y at each value of X

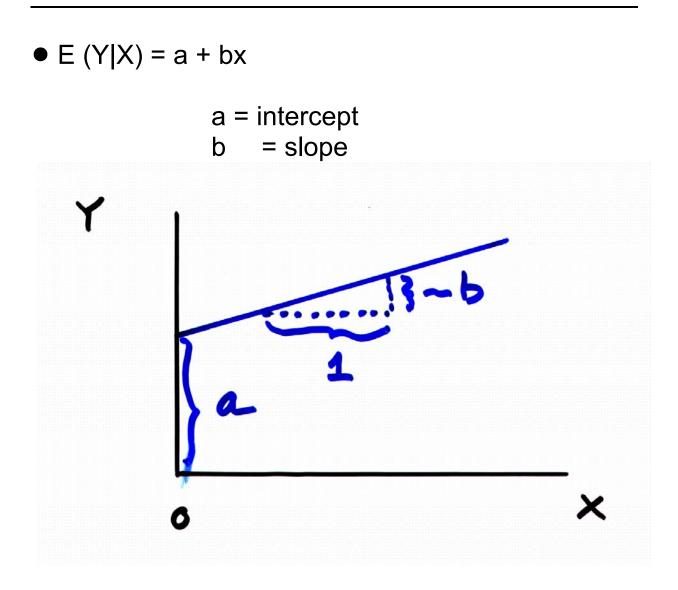
- eg, Regression of salary on age (and, possibly, other Xs)
- Notational convention -- express as either:





• E (Y|X) is a *linear* function of X

3.1 Simple linear regression (SLR)



• E (Y|X) = $\beta_0 + \beta_1 X$

May use either (a, b) or (β₀, β₁) -- arbitrary notation!

3.2 Multiple linear regression (MLR)

In simple linear regression (SLR)
 One X
 E (Y | X) = β₀ + β₁X
 In Multiple linear regression (MLR)
 More than one X

$$E (Y | X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots + \beta_p X_p$$
$$= \sum_{j=0}^{p} \beta_j X_j$$

• Salary =
$$\beta_0 + \beta_1$$
 Age + β_2 Grad Date

Units are (\$1,000) = (\$1,000) + (\$1,000/Yr) * (Yr) + (\$1,000/Yr) * (Yr)

3.3 MLR with two predictors

MLR model

 $Y_{i} = E(Y_{i} | X_{i1}, X_{i2}) + \varepsilon_{i}$ $= \beta_{0} + \beta_{1}X_{i1} + \beta_{2}X_{i2} + \varepsilon_{i}$

or,

Response = Prediction using Xs + Error

The errors ε_1 , ε_2 , ..., ε_n are independent and identically distributed (iid) normal variates, mean 0, variance σ^2 (unknown constant); or, more briefly, ε_i iid N(0, σ^2), i=1,...,n

3.4 Interpretation of MLR coefficient

- β₁ is the expected change in Y when X₁ increases one unit and X₂ remains fixed; or
- β₁ is the difference between average Ys for two populations that differ in X₁ by one unit and have the same X₂
- Example of a regression equation

$$Y = \beta_0 + \beta_1 (Age - 40) + \beta_2 Gender + \epsilon$$

Salary = 50 + 1 (Age - 40) - 3 Gender + ε

Salary in \$1,000s, Age in years and Gender = 0 if male and 1 if female

What is the average salary for 50 year old males?

$$Ave(Y) = 50 + 1 (50-40) - 3(0) = $60K$$



- β_0 average salary (\$1,000) for a 40 year old male
- $\label{eq:beta_linear} \begin{array}{l} & \beta_1 \mbox{ increase in average salary for every year} \\ & \mbox{ of age for a given gender} \\ & (\mbox{ M or F}) \end{array}$
- $-\beta_2$ difference in average salary for women vs. men of the same age
- Consider observations for two people and subtract expected responses:

 (Y_1, X_1, X_2) and $(Y_2, (X_1 + 1), X_2)$

EY ₂	=	$\beta_0 + \beta_1(X_1 + 1) + \beta_2 X_2$
– EY ₁	=	$\beta_0 + \beta_1(X_1) + \beta_2 X_2$

$E(Y_{2}-Y_{1})$	=	$0 + \beta_1$	+ 0
	=	β ₁	

 β₁ is the expected difference in Y corresponding to a unit difference in X₁ given X₂ is the same

4.1 Data and Question

The data come from the book by Pagano and Gauvreau: *Principles of Biostatistics*

Y -	Average expenditure (\$s) per admission
X ₁ -	Average length of stay (days)
X ₂ -	Average employee salary (\$s)
n = 51;	50 states + DC
Scientifi	c Question:

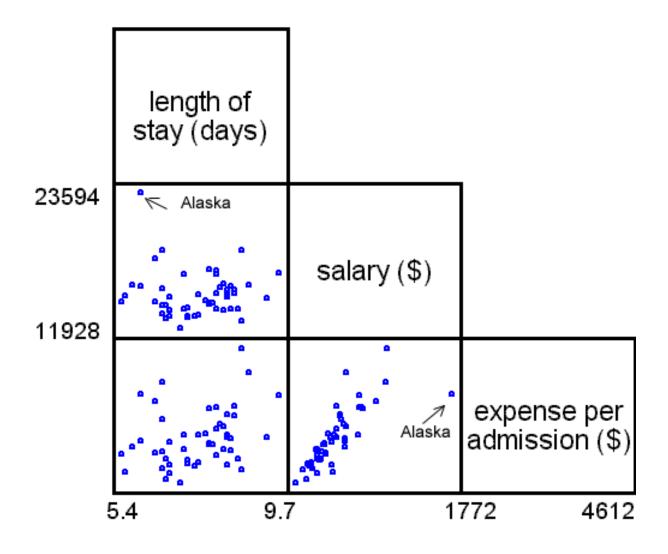
How is per capita expenditure related to:

— Length of stay

— Employee salary

4.2 Display the Data

• Use a scatterplot matrix



4.3 Model and interpretation of coefficients

• MLR model

$$\mathbf{Y} = \beta_0 + \beta_1 \mathbf{X}_1 + \beta_2 \mathbf{X}_2 + \epsilon$$

where

- **Y** Expenditure per admission in \$s
- X_1 LOS in days
- **X**₂ Salary in \$s

4.3 Model and interpretation of coefficients Interpretation of coefficients

 β_0 - expected per capita expenditure when LOS = 0 and salary = 0;

Nonsense! But OK- don't need β_0 to answer the questions posed

- β_1 difference in expected per capita expenditure (\$s) for two states with same average salary but LOS that differs by one day
- β_2 difference in expected per capita expenditure (\$s) for two states with same average LOS but salary that differs by one dollar

4.4 Results and interpretation

• Stata code

regress expadm los salary

• Stata log

	o. MLR: expense expadm los sa	-	ission on a	salary and	los	
•	SS				Number of obs	= 51
Model Residual	13840987.8 4396616.24	2 6920 48 9159	493.90 6.1716		F(2, 48) Prob > F R-squared Adj R-squared	= 0.0000 = 0.7589
Total	18237604.0	50 3647	52.081		Root MSE	= 302.65
expadm		Std. Err.	t	P> t	[95% Conf.	
salary	213.7967 .248994 -2582.736	.0217992	11.422	0.000	128.9325 .2051638 -3517.219	.2928241

4.4 Results and interpretation

• Summary of results

Variable	Ŕ	SE _ŝ	t	95% CI
Intercept	-2,583	465	-5.6	
LOS (days)	214	42.2	5.1	(129, 299)
Salary (\$s)	0.249	0.022	11.4	(0.205, 0.293)

Interpretation

- We estimate that expected expenditure per admission will be \$214 higher (95% CI: \$129-299) in a state whose average LOS is one day longer than another state with the same average employee salary
- For two states with the same LOS, we estimate an additional average expenditure per admission of \$249 (95%: 205-293) by a state whose salary is \$1,000 higher

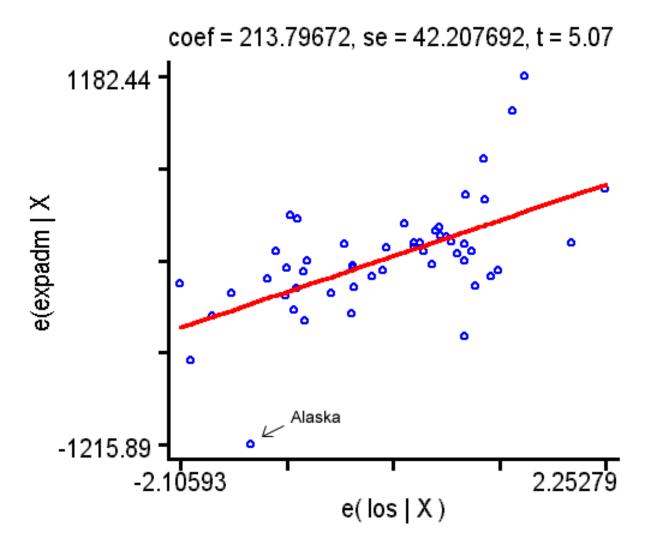
5.1 Added variables plots

 Check for curvature or other unusual patterns or unusual points

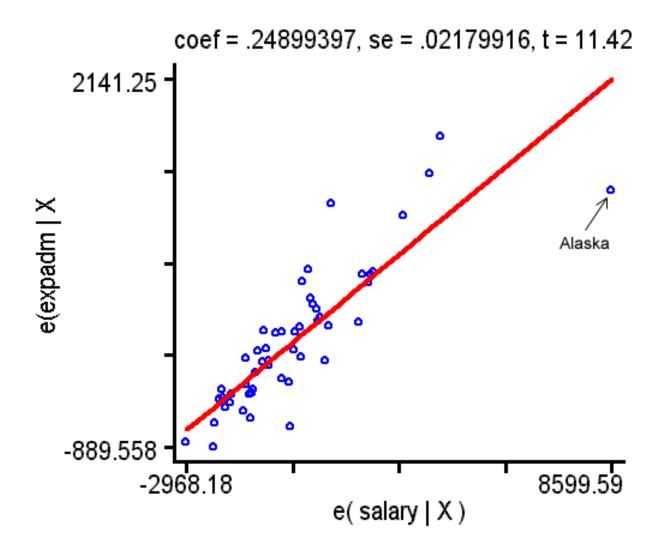
Stata commands

regress expadm los salary avplot los avplot salary

Stata AVP graphs



5.1 Added variables plots



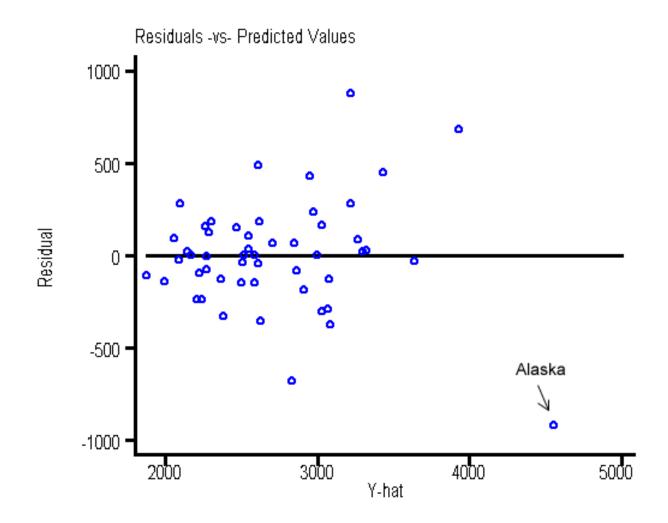
 Clearly, one point is unusual with respect to the others – it is data for the state of "Alaska"

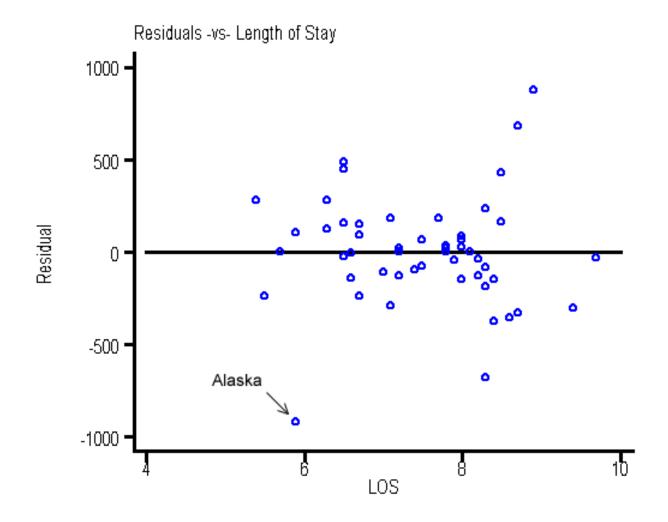
- Plot residuals -vs- Ý
- Plot residuals -vs- each X or, perhaps, -vs- a new X, not in the current model
- Look for departures from random patterns within each "bin" of *X*s or \hat{Y}
- Look for curvature, unequal variance, points that are outliers, or other unusual patterns or points
- Stata Code

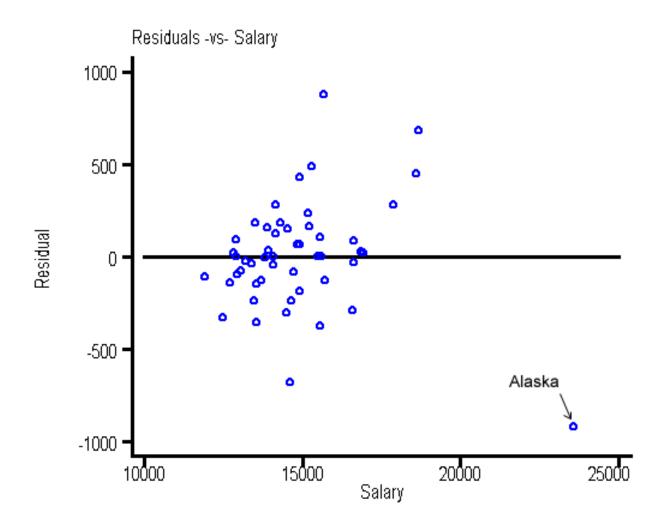
(Assumes fit already done)

predict yhat predict e, residuals

graph e yhat graph e los graph e salary







• CONCLUDE:

Exclude "Alaska," an unusual point (outlier) and re-fit the model

Determine sensitivity of results to the unusual point discovered above by comparing models with and without "Alaska"

• Stata log

* Since fit will not accept strings in if statements (UGH),							
* must	* must convert state(string variable) to stateno (numeric variable)						
encode stat	encode state , gen(stateno)						
* Check list state	* Check list state stateno , nolabel						
regress exp	oadm los sala	ry if staten	no ~= 2				
	SS				Number of obs =		
-	14521751 0				F(2, 47) =		
	14531751.2				Prob > F =		
	2849649.30				R-squared = Adj R-squared =		
	17381400.5						
expadm	Coef.	Std. Err.	t	P> t	[95% Conf. I	nterval]	
los	168.6216	35.48546	4.752	0.000	97.23405	240.0091	
salary	.3239785	.0231285	14.008	0.000	.27745	.3705069	
_cons	-3325.111		-8.196		-4141.258 -2	2508.965	

Sensitivity of results to including/excluding the extreme point (Alaska)

	with Alaska				w/o Alaska	a
	ó			Ó		
Var		Se _ĝ	t	β	Se _ĝ	t
Intercpt	-2583	465	-5.6	-3325	406	-8.2
LOS	214	42	5.1	168	35	4.7
Salary	0.249	.022	11.4	0.323	.023	14.0

Var	DBETA	DBETAS
LOS	214-168 = 46	(214-168) / 42 = 1.1
Salary	.249323 =074	(249323) / .022 = -3.4

• DBETA , calculated above, is the difference between $\hat{\beta}$ (with the outlier) minus $\hat{\beta}$ (without the outlier) – it directly indicates the "influence" of the unusual point on the effect estimate

• DBETAS. calculated above, is a standardized version DBETA -- divide DBETA by $se_{\hat{\beta}}$ (from the model including all points)

These DBETAS will, approximately, have mean 0 and SD of 1

Values of | DBETAS | greater than 2 only occur by chance with probability about 0.05; values greater than 3 occur by chance with probability about 0.003

 Stata calculates DBETAS using a more accurate standardizing formula, so results will not exactly agree with those done by hand

• Stata log

(Note Stata's DBETAS are more extreme than hand calculated values shown above)

* Determine and list DBETA for the outlier * Fit model with all the data points regress expadm los salary dfbeta los salary DFlos: DFbeta(los) DFsalary: DFbeta(salary) list state DF* if stateno == 2 state DFlos DFsalary 48. Alaska 1.315527 -4.227888

Check for non-linearity in the dependence of expenditures on LOS and salary

- Use the "broken-arrow" (linear spline) MLR to model the non-linearity
- Drop Alaska from the dataset for this analysis
- While the AVP and residual plots, showed no obvious non-linearity, fit a spline model with two slopes for LOS (breaking at 7 days) and two slopes for salary (breaking at \$15,000) to check, more formally, whether the fit can be improved by extending the model with non-linear terms

Define

$$L1 = (LOS - 7)^{+}$$
$$= \begin{cases} 0 & \text{if } LOS \le 7\\ LOS - 7 & \text{if } LOS > 7 \end{cases}$$

SAL1 = $(Salary - 15,000)^+$

1.
$$Y = \beta_0 + \beta_1(LOS-7) + \beta_2(Salary-15,000)$$

+ $\beta_3(LOS-7)^+$ + ϵ

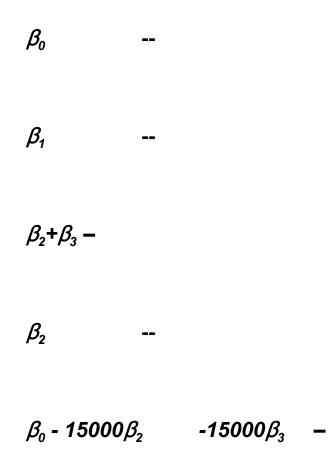
Interpretation of coefficients :

eta_{o}	-	Avera	LOS i \$15,0	is 7 da 00 i line be	ture when the ays and salary is ntercept for the efore the break
β_1	-	Rate	with I	LOS d	n expenditures ay 7 or before salary
$\beta_1 + \beta_3$	-	Rate	with I		n expenditures fter day 7 for a y
β_2	-	Rate		1000 c	n expenditures of salary for a
β ₀ - 7	β ₁ -7£	3 ₃		Interc	cept for the LOS line after the 7 day break point if the salary is \$15,000

2.
$$Y = \beta_0 + \beta_1 (LOS-7) + \beta_2 (Salary-15,000)$$

+ β_3 (Salary-15,000)⁺ + ϵ

Interpretation of coefficients -- you do!:



Stata log

* Fit linear splines for LOS and Salary -- in separate models, w/o Alaska * Center los and salary at their means (excluding Alaska) egen losc = mean(los) if stateno~=2 replace losc = los - losc salaryc = mean(salary) if stateno~=2 eaen replace salaryc = salary - salaryc * Generate non-linear terms gen 11 = (los-7) * (los>7) gen sal1 = (salary - 15000) * (salary>15000) * Fit models regress expadm losc salaryc 11 if stateno ~= 2 Source | SS df Number of obs = 50 MS F(3, 46) = 78.48Model | 14540359.3 3 4846786.43 Residual | 2841041.19 46 61761.765 Prob > F = 0.0000R-squared = 0.8365 Adj R-squared = 0.8259 Total | 17381400.5 49 354722.459 Root MSE = 248.52_____ expadm | Coef. Std. Err. t P>|t| [95% Conf. Interval] _____+____
 losc
 206.6644
 108.0117
 1.913
 0.062
 -10.75183
 424.0806

 salaryc
 .3267898
 .0245278
 13.323
 0.000
 .277418
 .3761616

 11
 -57.01222
 152.7122
 -0.373
 0.711
 -364.406
 250.3815

 cons
 2740.327
 117.4716
 23.328
 0.000
 2503.869
 2976.785
 regress expadm losc salaryc sal1 if stateno ~= 2 MS Source | SS df Number of obs = 50 F(3, 46) = 78.63______ Model | 14544925.1 3 4848308.36 Residual | 2836475.39 46 61662.5084 Prob > F = 0.0000R-squared = 0.8368 Adj R-squared = 0.8262 Total | 17381400.5 49 354722.459 Root MSE = 248.32 _____ Coef. Std. Err. t P>|t| expadm | [95% Conf. Interval] losc170.484736.012394.7340.00097.99555242.9739salaryc.3048594.04748676.4200.000.2092736.4004452sal1.0360975.07809630.4620.646-.1211021.1932971_cons2681.58950.6827552.9090.0002579.5692783.608 ____

Summary of results

	Variable	β	Se _ŝ	t
--	----------	---	-----------------	---

Intercept	2740	74	37
LOS-7	206.7	108.0	1.91
L1	-57.0	152.7	-0.37
Salary-15K	0.33	.024	13
	_		
Variable	β́	se _ŝ	t
Intercept	2682	66	41
LOS-7	170.5	36	4.7
Salary-15K	0.30	0.047	6.4
SAL1	0.036	0.078	0.46

What have we learned about non-linearity?

- Neither L1 nor SAL1 improves model substantially
- Therefore, the dependence of expenditures on either LOS or salary is adequately approximated by a linear function

5.5 Interaction

 Whether LOS is an "effect modifier" for salary or, equivalently, is there an "interaction" between salary and LOS depends on the answer to either of the following two questions:

Is the effect of salary on expenditures different at different levels of LOS?

or, equivalently,

Is the effect of LOS on expenditures different at different levels of salary?

• Once again, leave "Alaska" out of the analysis

 Extend the model with an interaction term that is the product of LOS and salary (LOS*Salary)

$\mathbf{Y} = \beta_0 + \beta_1 (LOS-7) + \beta_2 (Salary-15,000)/1000$

+ β_3 (LOS-7)*(Salary-15,000)/1000 + ϵ

5.5 Interaction

Stata log

* Fit model with interaction for LOS*Salary w/o Alaska * * Scale salary, divide by 1000 replace salaryc = salaryc/1000 * Generate interaction term gen losXsal = losc * salaryc * Fit model regress expadm losc salaryc losXsal if stateno~=2 Source | SS df MS Number of obs = 50 $\begin{array}{rcl} \text{Kullber of obs} &=& 30\\ \text{F(3, 46)} &=& 83.27\\ \text{Prob} > \text{F} &=& 0.0000\\ \text{R-squared} &=& 0.8445 \end{array}$ _____ Model | 14678583.7 3 4892861.22 Residual | 2702816.82 46 58756.8873 Adj R-squared = 0.8344 = 242.40 Total | 17381400.5 49 354722.459 Root MSE _____ expadm | Coef. Std. Err. t P>|t| [95% Conf. Interval]

 losc |
 153.6725
 36.19012
 4.246
 0.000
 80.82559
 226.5195

 salaryc |
 326.8532
 22.84075
 14.310
 0.000
 280.8772
 372.8293

 losXsal |
 38.3008
 24.22849
 1.581
 0.121
 -10.46861
 87.07022

 _cons |
 2691.143
 34.59304
 77.794
 0.000
 2621.511
 2760.775

Summary of results

Variable	β	se _ŝ	t
		•	
Intercept	2691.1	34.5	77.8
(LOS-7)	153.7	36.2	4.2
(Salary-15K)/1000	326.9	22.8	14.3
(LOS-7)* (Salary-15K) /1000	38.3	24.2	1.58

5.5 Interaction

• Interpretation of $\hat{\beta}_3 = 38.3$?

LOS = 8 (or
$$X_1 + 1$$
)
 $\hat{Y} = \hat{\beta_0} + \hat{\beta_1}(8-7) + \hat{\beta_2}X_2 + \hat{\beta_3}(8-7)X_2$
 $= \hat{\beta_0} + \hat{\beta_1} + (\hat{\beta_2} + \hat{\beta_3})X_2$

LOS = 7 (or
$$X_1$$
)
 $\hat{Y} = \hat{\beta_0} + \hat{\beta_1}(7-7) + \hat{\beta_2}X_2 + \hat{\beta_3}(7-7)X_2$
 $= \hat{\beta_0} + \hat{\beta_2}X_2$

If LOS=7:

Average expenditures per capita are

\$326 higher per \$1,000 additional salary

but if *LOS* = *8*,

Average expenditures per capita are

\$326 + \$38.3 = \$364.3 higher per \$1,000 additional salary

• But, only weak/moderate evidence that this interaction is not 0

6. Scientific Summary

- Our multiple linear regression analysis shows that a state's expenditures per admission is statistically significantly associated with both LOS and employee salary (Model with two covariates)
- LOS being equal, a state whose average employee salary was \$1,000 higher averaged \$323 more per admission(95% CI: \$277, \$371) than another state
- With equal average salaries, a state with 1 day shorter LOS averaged \$168 less per admission (95% CI: \$97, \$240)
- The variation in expenditures per admission among states with comparable LOS and salary was ± ô = \$246

(from root MSE on Stata output)

 The above statements relate to 49 states + DC, excluding Alaska, using the model in Section 4.3

6. Scientific Summary

- There is little evidence that the dependence of expenditures on LOS and salary is nonlinear or that these two predictors interact with one another
- The reason Alaska was excluded is due to the unusually high employee salaries and low expenditures per admission caused by the use of state oil profits to subsidize hospital costs.

If the state Alaska is included in the data, the estimates and 95% CI are somewhat different – you do!

• The do-file and Stata dataset for the examples in this lecture are posted on the course website on the "Classes" page:

cl4ex1.do

hospital.dta

• Listing of do-file script:

- version 7.0
- * MLR.DO MLR
- * Example: Hospital Cost Data from Pagano and Gauvreau
- * Stata dataset: hospital.dta
- * Assumes files are in folder: [path]\bio623
- * To run this program, use the following Stata commands:
- * cd [path]\bio623 ... change directory to folder bio623
- * do MLR
- * OUTLINE:
- * Part a. Scatterplot matrix
- * Part b. MLR: expense per admission on salary and los
- * Part c. Added Variable plots -- uses regress above
- * Part d. MLR: Plot residuals vs Xs and predicteds for model checking
- * Part e. Re-fit without the outlier "Alaska"
- * Part f. Determine and list DBETA for the outlier
- * Part g. Fit linear splines for LOS and Salary -- in separate models,
 * w/o Alaska
- * Part h. Fit model with interaction for LOS*Salary
 * w/o Alaska

```
* Housekeeping
* Clear work space
clear
* Turn off -more- pause
set more off
* Set directory for do and log files or use "cd" to get to it
* cd [path]\regress
* Save log file on disk, use .txt so Notepad will open it
capture log close
log using cl4ex1.log, replace
* Make sub-folder to store graph images
shell md cl4ex1
* Extend linesize for log
set linesize 100
* Access Stata dataset
use hospital, clear
* Dataset contents
describe
* Get variables, codes, descriptive stats
codebook los salary expadm
* Get stats, excluding Alaska
summarize los salary expadm if state~="Alaska"
* List 5 records for checking
list in 1/5
```

* Part a. Scatterplot matrix
* (Increase textsize)
set textsize 150
graph los salary expadm, matrix half
set textsize 100

* Save the graph on disk in ps1-1 folder as Windows metafile fig1.wmf gphprint , saving(cl4ex1\fig1.wmf,replace)

* Part b. MLR: expense per admission on salary and los regress expadm los salary

```
* Part c. Added Variable plots -- uses regress above
set textsize 150
avplot los , l1(" ") l2("e(expadm | X")
gphprint , saving(cl4ex1\fig2a.wmf,replace)
avplot salary , l1(" ") l2("e(expadm | X")
gphprint , saving(cl4ex1\fig2b.wmf,replace)
set textsize 100
```

* Part d. MLR: Plot residuals vs Xs and predicteds for model checking
*
*
*

```
predict yhat
predict e, residuals
* Make the plots
graph e yhat , yline(0) xlab ylab t1("Residuals -vs- Predicted Values") 11("
") 12("Residual") b1(" ") b2("Y-hat")
gphprint , saving(cl4ex1\fig3a.wmf,replace)
graph e los , yline(0) xlab ylab t1("Residuals -vs- Length of Stay") 11(" ")
12("Residual") b1(" ") b2("LOS")
gphprint , saving(cl4ex1\fig3b.wmf,replace)
graph e salary , yline(0) xlab ylab t1("Residuals -vs- Salary") 11(" ")
12("Residual") b1(" ") b2("Salary")
gphprint , saving(cl4ex1\fig3b.wmf,replace)
* Part e. Re-fit without the outlier "Alaska"
* Since fit will not accept strings in if statements (UGH),
      must convert state(string variable) to stateno (numeric variable)
encode state , gen(stateno)
* Check
list state stateno , nolabel
regress expadm los salary if stateno ~= 2
* Part f. Determine and list DBETA for the outlier
* Fit model with all the data points
regress expadm los salary
```

```
dfbeta los salary
list state DF* if stateno == 2
* Part g. Fit linear splines for LOS and Salary -- in separate models,
*
       w/o Alaska
* Center los and salary at their means (excluding Alaska)
egen
       losc = mean(los)
                          if stateno~=2
replace losc = los - losc
egen
       salaryc = mean(salary)
                               if stateno~=2
replace salaryc = salary - salaryc
* Generate non-linear terms
gen 11 = (10s-7) * (10s>7)
gen sal1 = (salary - 15000) * (salary>15000)
* Fit models
regress expadm losc salaryc 11 if stateno ~= 2
regress expadm losc salaryc sal1 if stateno ~= 2
* Part h. Fit model with interaction for LOS*Salary
       w/o Alaska
* Scale salary, divide by 1000
replace salaryc = salaryc/1000
* Generate interaction term
gen losXsal = losc * salaryc
```

JHU Graduate Summer Institute of Epidemiology and Biostatistics, June 16- June 27, 2003 Materials extracted from: Biostatistics 623 © 2002 by JHU Biostatistics Dept.

* Fit model

regress expadm losc salaryc losXsal if stateno~=2

* Close log file -- Only once all errors have been fixed

*log close