ANOVA - Non-Parametric Methods

ANOVA, still

$\{Y_{ti}\}$ independent with $Y_{ti} \sim \text{Normal}(\mu_t, \sigma^2)$ for t = 1 . . . k.

Test $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$

The usual statistic:

$$F = M_B/M_W = \frac{\sum_t n_t\,(\bar{Y}_{t\cdot} - \bar{Y}_{\cdot\cdot})^2/(k-1)}{\sum_t \sum_i (Y_{ti} - \bar{Y}_{t\cdot})^2/(\sum_t n_t - k)}$$

P-values:

(a) Use the $F(k-1, \sum n_t - k)$ distribution.

(b) Use a permutation test.

Assumptions:

(a) Underlying dist'ns are normal with common variance.

(b) Underlying dist'ns are the same.

Non-parametric ANOVA

An alternative approach: the Kruskal-Wallis test.

Rank all of the observations from 1, 2, . . . , N.

Let R_{ti} = the rank for observation Y_{ti}.

Let $\bar{R}_{t\cdot} = \sum_i R_{ti}/n_t$ = the average rank for group t.

Null hypothesis, H_0: the underlying distributions are all the same.

$$E(\bar{R}_{t\cdot} \mid H_0) = \frac{N+1}{2}$$

$$SD(\bar{R}_{t\cdot} \mid H_0) = \sqrt{\frac{(N+1)\ (N-n_t)}{12\ n_t}}$$

Kruskal-Wallis test statistic

$$H = \sum_t \left(\frac{N-n_t}{N}\right) \times \left[\frac{\bar{R}_{t\cdot} - E(\bar{R}_{t\cdot} \mid H_0)}{SD(\bar{R}_{t\cdot} \mid H_0)}\right]^2$$

$$= \cdots = \frac{12}{N\,(N+1)} \sum_t n_t \left[\bar{R}_{t\cdot} - \left(\frac{N+1}{2}\right)\right]^2$$

Under H_0, and if the sample sizes are large, $H \sim \chi^2(df = k - 1)$.

$\longrightarrow$ Alternatively, we could use a permutation test to estimate a P-value.

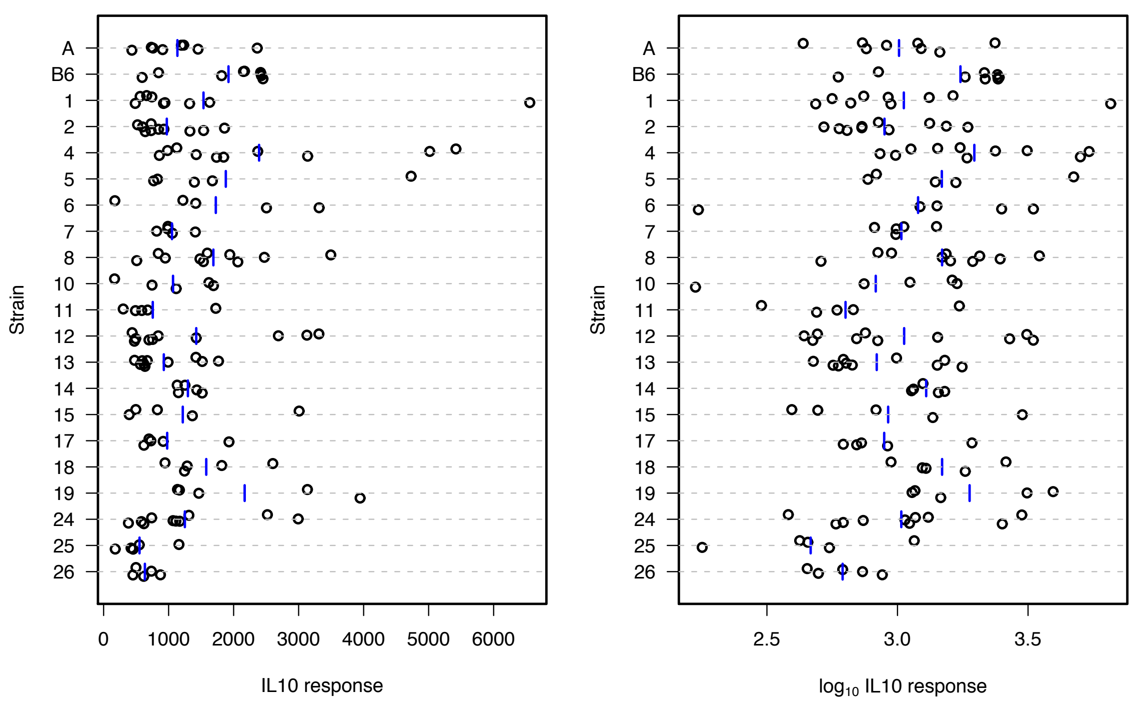
The function `kruskal.test()` in R will calculate the statistic.

Note

→ In the case of two groups, the Kruskal-Wallis test reduces exactly to the Wilcoxon rank-sum test.

→ This is just like how ANOVA with two groups is equivalent to the two-sample t test.

Example



ANOVA Tables

Original scale / 1000:

source	SS	df	MS	F	P-value
between strains	33	20	1.69	1.70	0.042
within strains	124	125	0.99		
total	157	145			

$\longrightarrow$ permutation P-value = 0.043

$\log_{10}$ scale:

source	SS	df	MS	F	P
between strains	3.35	20	0.167	2.25	0.0036
within strains	9.29	125	0.074		
total	12.63	145			

$\longrightarrow$ permutation P-value = 0.003

K-W results

The observed Kruskal-Wallis statistic for these data was 41.32.

$\longrightarrow$ Note that it doesn't matter whether you take logs!

Since there were 21 strains, we can compare this to a χ^2 distribution with 20 degrees of freedom. Thus we obtain the P-value = 0.003.

With a permutation test, I got $\hat{P}$ = 0.0015 (on the basis of 10,000 simulations.

In the case of ties...

In the case of ties, we assign the average rank to each.

Example:

A:		3.5		3.7		4.0	4.2	4.3	
B:					3.9			4.3	4.5
C:	3.1		3.6			4.0		4.3	
	(1)	(2)	(3)	(4)	(5)	(6/7)	(8)	(9/10/11)	(12)
						$\downarrow$		$\downarrow$	
						6.5		10	

Then we apply a correction factor.

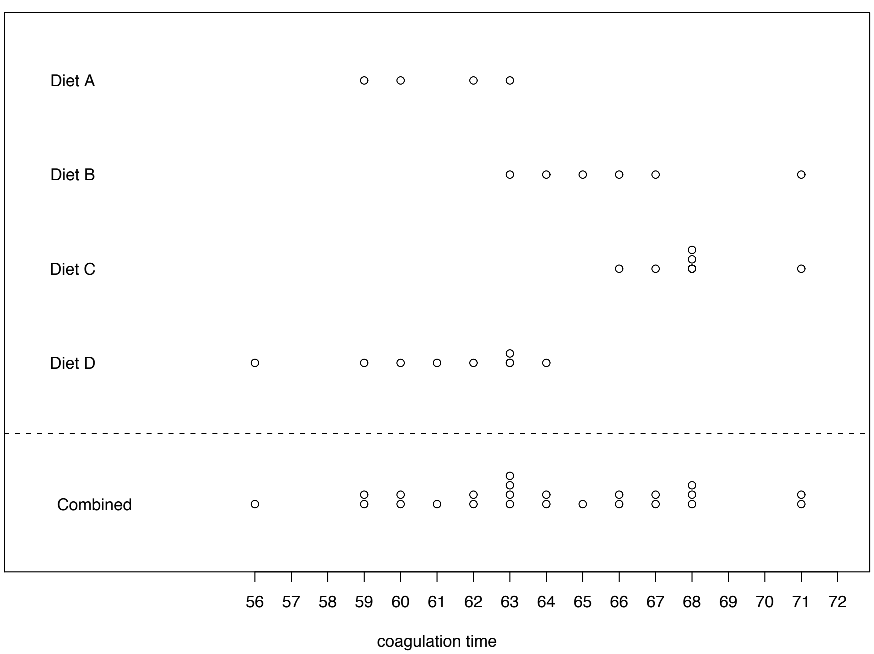
Let $N = \sum_t n_t$ and T_i = no. observations in the i^{th} set of ties (can be 1).

Let $D = 1 - \sum_i (T_i^3 - T_i)/(N^3 - N)$

Use the statistic $H' = H/D$.

Note that $D \leq 1$ and so $H' \geq H$. For the example, $D = 1 - \frac{(2^3-2)+(3^3-3)}{12^3-12} \approx 0.983$.

Blood coagulation time



Example (continued)

A	B	C	D	rank	avg rank
			56	1	1
59				2	2.5
			59	3	2.5
60				4	4.5
			60	5	4.5
			61	6	6
62				7	7.5
			62	8	7.5
63				9	10.5
	63			10	10.5
			63	11	10.5
			63	12	10.5
	64			13	13.5
			64	14	13.5
	65			15	15
	66			16	16.5
		66		17	16.5
	67			18	18.5
		67		19	18.5
		68		20	21
		68		21	21
		68		22	21
	71			23	23.5
		71		24	23.5

Example (continued)

A	62	60	63	59					61
	7.5	4.5	10.5	2.5					6.25
B	63	67	71	64	65	66			66
	10.5	18.5	23.5	13.5	15.0	16.5			16.25
C	68	66	71	67	68	68			68
	21.0	16.5	23.5	18.5	21.0	21.0			20.25
D	56	62	60	61	63	64	63	59	61
	1.0	7.5	4.5	6.0	10.5	13.5	10.5	2.5	7.00

Calculation of K-W test statistic

	A	B	C	D
n_t	4	6	6	8
$\bar{R}_{t\cdot}$	6.25	16.25	20.25	7.00

$N = 24$

$\frac{N+1}{2} = 12.5$

$$H = \frac{12}{N\,(N+1)} \sum_t n_t \left[\bar{R}_{t\cdot} - \left(\tfrac{N+1}{2}\right)\right]^2$$

$$= \frac{12}{24\times 25} \left\{4 \times (6.25 - 12.5)^2 + \cdots + 8 \times (7.00 - 12.5)^2\right\}$$

$$= 16.86$$

The ties: T_i = (1 2 2 1 2 4 2 1 2 2 3 2)

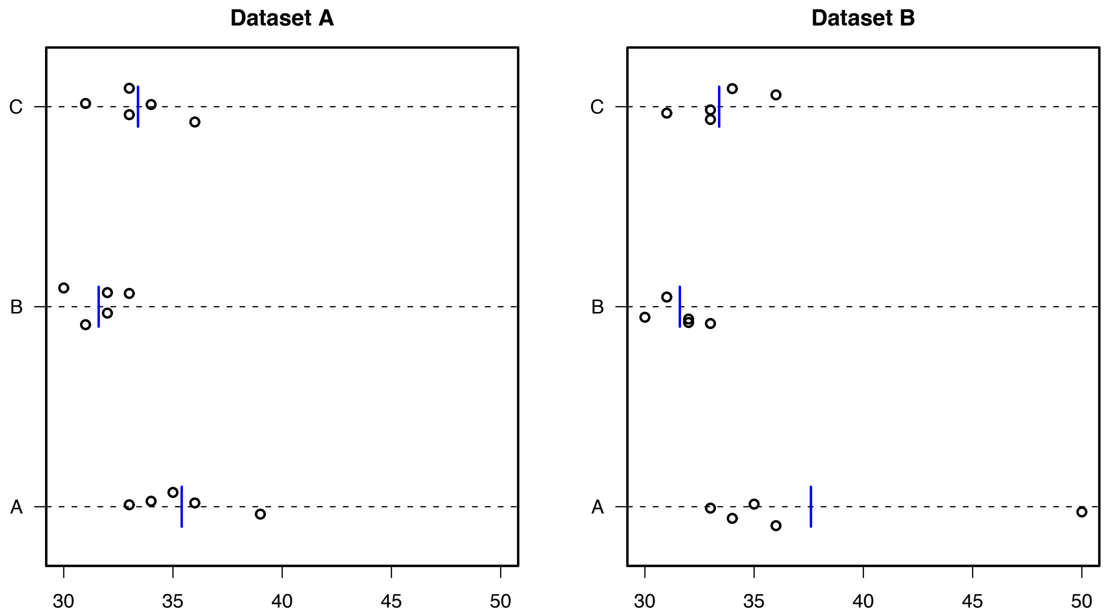
$$D = 1 - \sum_i (T_i^3 - T_i)/(N^3 - N) = \ldots = 0.991$$

$H' = H/D = 16.86 / 0.991 = 17.02$ (df = 3) $\longrightarrow$ P-value ≈ 0.0007

A few points

- Calculation of P-values: (avoiding type I errors)
  - F statistic: F distribution (requires normality)
  - K-W statistic: χ^2 distribution (requires large samples)
  - Either statistic: Permutation tests
- Power: (avoiding type II errors)
  - K-W statistic more resistant to outliers
  - F statistic more powerful in the case of normality
- K-W statistic: don't need to worry about transformations.

A fake example



Results

Dataset	Method	Statistic	nominal P-value	Permu'n P-value
A	ANOVA	5.48	0.020	0.017
	K-W	7.64	0.022	0.012
B	ANOVA	2.64	0.112	0.023
	K-W	7.64	0.022	0.012

Distributions