

# Maximum Likelihood Estimation

## Estimation

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Goal: Estimate a population parameter  $\theta$ .

Data:  $X_1, X_2, \dots, X_n \sim \text{iid}$  with distribution depending on  $\theta$ .

If one has many estimators to choose from, pick

- That with the smallest SE, among all unbiased estimators.
- That with the smallest RMS error, even if biased.

→ Sometimes it is not clear how to form even one good estimator.

# Maximum likelihood estimation

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Likelihood function:  $L(\theta) = \Pr(\text{data} \mid \theta)$

Log likelihood:  $l(\theta) = \log \Pr(\text{data} \mid \theta)$

## Maximum likelihood estimate:

→ Choose, as the estimate of  $\theta$ , the value of  $\theta$  for which the likelihood function  $L(\theta)$  (or equivalently, the log likelihood function) is maximized.

You need to solve these equations analytically or numerically!

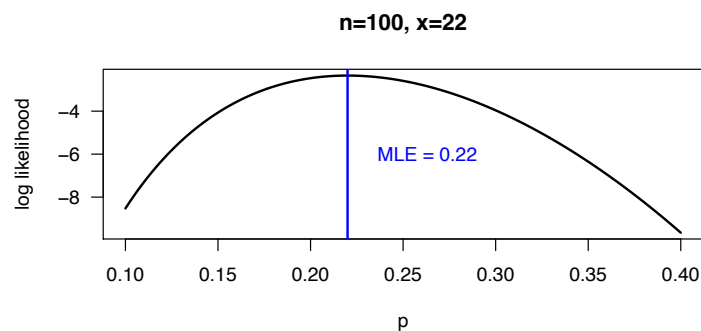
## Example 1

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Suppose  $X \sim \text{Binomial}(n, p)$ .

log likelihood function:  $l(p) = \log \left\{ \binom{n}{x} p^x (1-p)^{(n-x)} \right\}$   
 $= x \log(p) + (n-x) \log(1-p) + \text{constant}$

MLE: the obvious thing:  $\hat{p} = x/n$

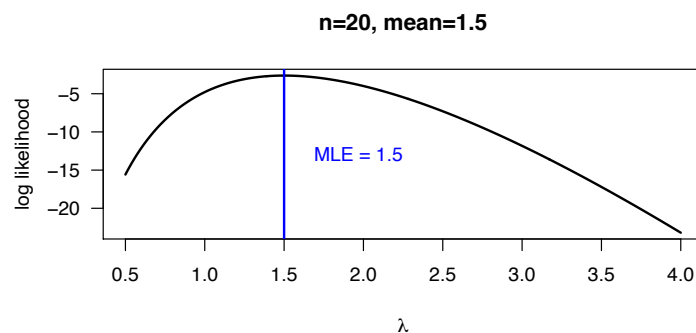


## Example 2

Suppose  $X_1, \dots, X_{20} \sim \text{iid Poisson}(\lambda)$ .

$$\begin{aligned} \text{log likelihood function: } l(\lambda) &= \log \left\{ \prod_i e^{-\lambda} \lambda^{x_i} / x_i! \right\} \\ &= \dots = -20\lambda + \left( \sum x_i \right) \log \lambda + \text{constant} \end{aligned}$$

MLE: the obvious thing:  $\hat{\lambda} = \bar{x}$



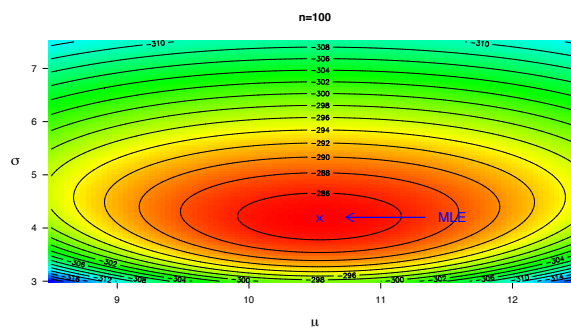
## Example 3

Suppose  $X_1, \dots, X_n \sim \text{iid } N(\mu, \sigma)$

$$\text{log likelihood function: } l(\mu, \sigma) = \log \left\{ \prod_i \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x_i - \mu}{\sigma} \right)^2 \right] \right\}$$

MLEs: almost the obvious things:

$$\hat{\mu} = \bar{x} \quad \hat{\sigma} = \sqrt{\sum (x_i - \bar{x})^2 / n}$$



## About MLEs

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Maximum likelihood estimation is a general procedure for finding a reasonable estimator

- In many cases, the MLE turns out to be the obvious thing.
- MLEs are often very good (but not necessarily the best) possible estimators:
  - Unbiased or nearly unbiased.
  - Small standard errors.
- Sometimes obtaining the MLEs requires hefty computation!

## Example 4: ABO blood groups

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Phenotype	Genotype	Frequency
O	OO	$p_O^2$
A	AA or AO	$p_A^2 + 2p_Ap_O$
B	BB or BO	$p_B^2 + 2p_Bp_O$
AB	AB	$2p_Ap_B$

Frequencies under the assumption of Hardy-Weinberg equilibrium.

## Example 4: Data

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Phenotype	No. subjects	% subjects
O	117	46.8%
A	98	39.2%
B	29	11.6%
AB	6	2.4%
<b>Total</b>	<b>250</b>	<b>100%</b>

→ What are the estimates of  $p_A$ ,  $p_B$ ,  $p_O$ ?

## Example 4: Estimates

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Simple estimates:

$$\rightarrow \tilde{p}_O = \sqrt{0.468} = 0.684$$

$$\rightarrow \tilde{p}_A^2 + 2\tilde{p}_A 0.684 = 0.392 \rightarrow \tilde{p}_A = 0.243$$

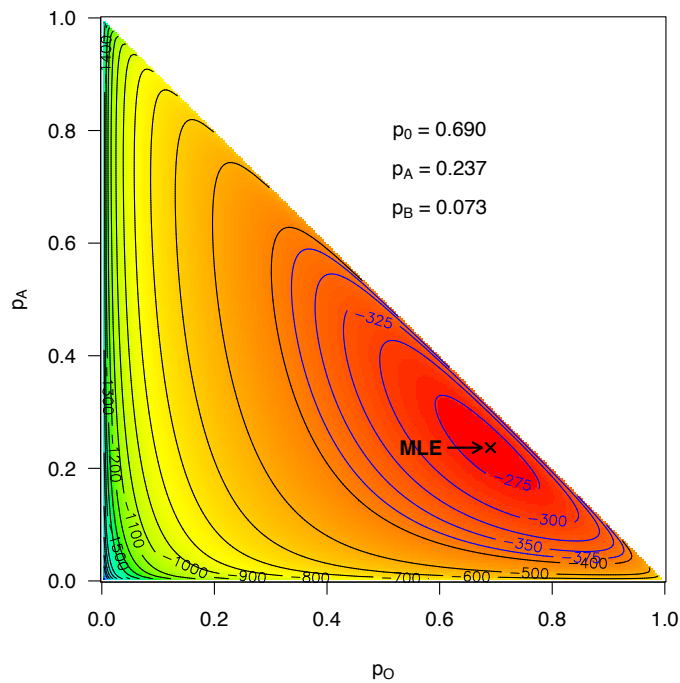
$$\rightarrow \tilde{p}_B = 0.024 / (2\tilde{p}_A) = 0.072$$

Log likelihood Remember the Multinomial distribution function!

$$l(p_O, p_A, p_B) =$$

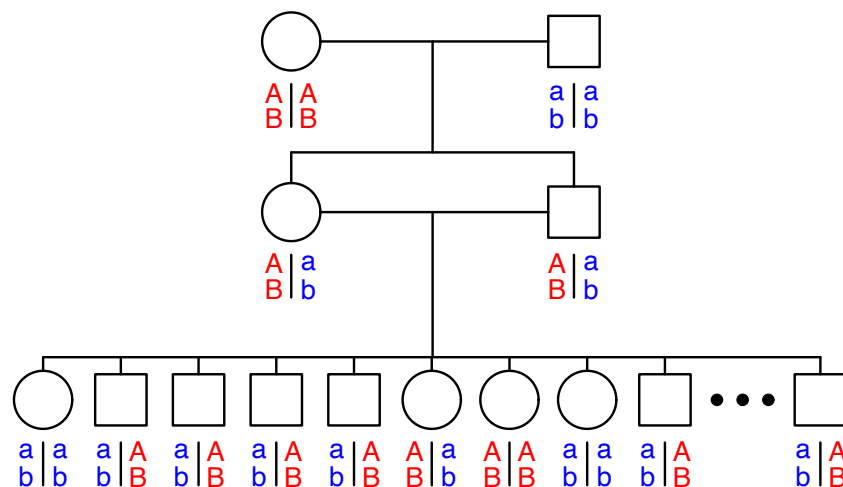
$$117 \log(p_O^2) + 98 \log(p_A^2 + 2p_A p_O) + 29 \log(p_B^2 + 2p_B p_O) + 6 \log(2p_A p_B)$$

## Example 4: log likelihood



## Example 5

Consider the problem of estimating the recombination fraction (call that parameter  $\theta$ ) between two genetic markers in an intercross.



→ Note: We won't observe the haplotypes.

## Example 5

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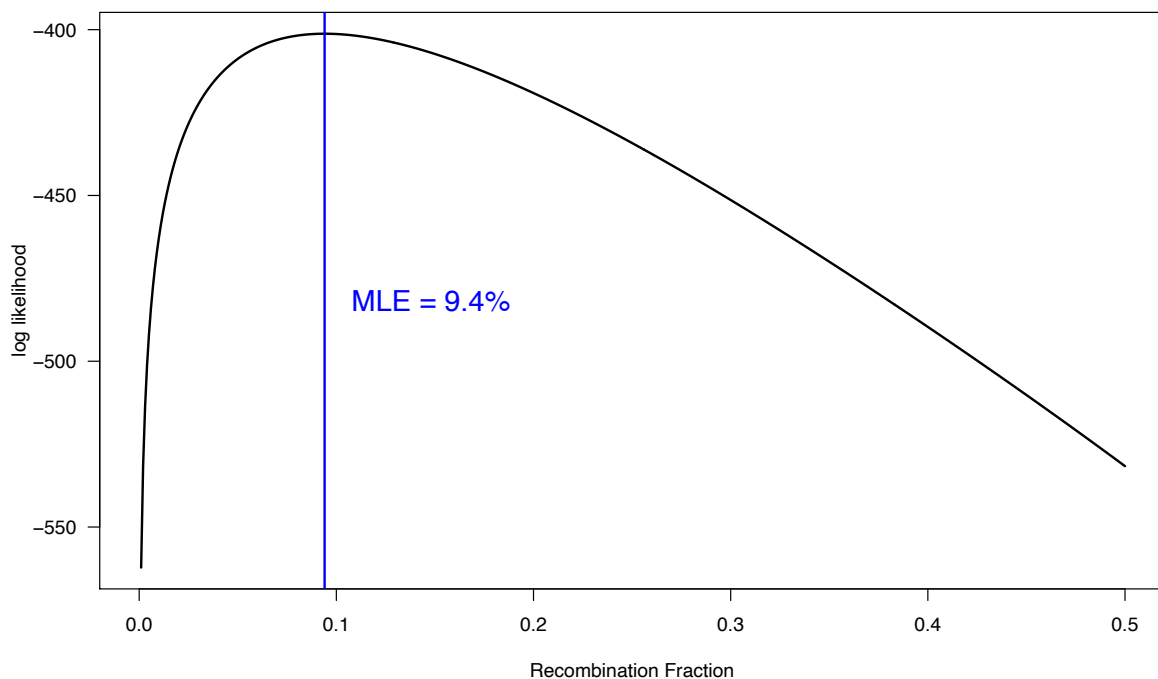
	Data			Probabilities		
	AA	Aa	aa	AA	Aa	aa
BB	58	9	0	$\frac{1}{4} (1 - \theta)^2$	$\frac{1}{2} \theta(1 - \theta)$	$\frac{1}{4} \theta^2$
Bb	8	95	14	$\frac{1}{2} \theta(1 - \theta)$	$\frac{1}{2} [\theta^2 + (1 - \theta)^2]$	$\frac{1}{2} \theta(1 - \theta)$
bb	1	12	53	$\frac{1}{4} \theta^2$	$\frac{1}{2} \theta(1 - \theta)$	$\frac{1}{4} (1 - \theta)^2$

→ Possible estimates of the recombination fraction,  $\theta$ ?

$$L(\theta) \propto \left\{ \frac{1}{4} (1 - \theta)^2 \right\}^{(58+53)} \times \left\{ \frac{1}{2} \theta(1 - \theta) \right\}^{(9+8+14+12)} \times \left\{ \frac{1}{4} \theta^2 \right\}^{(1+0)} \times \left\{ \frac{1}{2} [\theta^2 + (1 - \theta)^2] \right\}^{95}$$

## Example 5: log likelihood function

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## A closer view

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