Practice Problems (Maximum Likelihood Estimation)

- Suppose we randomly sample 100 mosquitoes at a study site, and find that 44 carry a parasite. Derive the maximum likelihood estimate for the proportion of infected mosquitoes in the population.

  Solution:

  The distribution function for a Binomial(n,p) is $P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$. In our case, $n=100$, and we observe $x=44$. The likelihood is a function of the unknown parameter p, given the data. Hence

  $$L(p)=\binom{100}{44}\times p^{44}\times(1-p)^{100-44}=\binom{100}{44}\times p^{44}\times(1-p)^{56}$$

  The log likelihood is

  $$l(p)=\log\left(\binom{100}{44}\times p^{44}\times(1-p)^{56}\right)=\log\binom{100}{44}+44\log(p)+56\log(1-p)$$

  To find the maximum of the log-likelihood, we take the first derivative of $l(p)$ with respect to p:

  $$l'(p)=0+44\frac{1}{p}+56\frac{1}{1-p}(-1)=\frac{44}{p}-\frac{56}{1-p}$$

  Setting the first derivative equal to zero yields

  $$0=\frac{44}{\hat{p}}-\frac{56}{1-\hat{p}}\quad\Longleftrightarrow\quad 0=44(1-\hat{p})-56\hat{p}\quad\Longleftrightarrow\quad 0=44-100\hat{p}\quad\Longleftrightarrow\quad \hat{p}=\frac{44}{100}$$

  Therefore, the maximum likelihood estimate for the proportion of infected mosquitoes in the population is 0.44.

- Suppose we take 3 random (independent) draws from a Poisson distribution, and obtain the numbers 18, 16, 23. Derive the maximum likelihood estimate for the Poisson parameter λ.

  Solution:

  The distribution function for a Poisson(λ) is $P(X=x)=\exp(-\lambda)\times\lambda^x/x!$. Since we have three independent draws, the probability of observing those data is

  $$\begin{aligned}&P(X_1=18\text{ and }X_2=16\text{ and }X_1=23)\\=\;&P(X=18)\times P(X=16)\times P(X=23)\\=\;&(\exp(-\lambda)\times\lambda^{18}/18!)\times(\exp(-\lambda)\times\lambda^{16}/16!)\times(\exp(-\lambda)\times\lambda^{23}/23!)\\=\;&\exp(-\lambda)^3\times\lambda^{18+16+23}/(18!\times16!\times23!)\\=\;&\exp(-\lambda)^3\times\lambda^{57}/(\text{some really large constant})\end{aligned}$$

  This is the likelihood function, and the only unknown quantity is λ. The log likelihood is

  $$\begin{aligned}l(\lambda)&=\log\left\{\exp(-\lambda)^3\times\lambda^{57}/(\text{some really large constant})\right\}\\&=-3\lambda+57\log(\lambda)-\log(\text{some really large constant}).\end{aligned}$$

To find the maximum of the log-likelihood, we take the first derivative of $l(\lambda)$ with respect to λ:

$$l'(\lambda) = -3 + 57/\lambda$$

Setting the first derivative equal to zero yields

$$0 = -3 + 57/\hat{\lambda} \quad \Longleftrightarrow \quad \hat{\lambda} = \frac{57}{3} = 19.$$

Therefore, the maximum likelihood estimate for the Poisson parameter λ is 19.