Practice Problems (Sample Size and Power)

- Assume that we have n independent measurements from a Normal distribution with unknown mean Δ and standard deviation 1, i.e. $X_1, \ldots, X_n \sim N(\Delta, \sigma = 1)$.

  1. What is the distribution of $\bar{X}$?
  2. To test for $H_0 : \Delta = 0$ versus $H_a : \Delta > 0$, what is your test statistic, and what is its distribution if H_0 was true?
  3. For which values of the test statistic do you reject the null hypothesis?
  4. If we want to have 80% power to detect a mean of $\Delta = 0.5$, how large must n be?

Solution:

1. $\bar{X} \sim N(\Delta, 1/\sqrt{n})$
2. $Z = \bar{X}/(1/\sqrt{n}) = \sqrt{n} \times \bar{X} \sim N(0,1)$ under H_0. Note that when $\Delta > 0, Z \sim N(\sqrt{n} \times \Delta, 1)$.
3. Since we have a one-sided test with the alternative $H_a : \Delta > 0$, we reject the null hypothesis only for large positive values. If we want to control the type I error rate at 5%, we use 1.64 (`qnorm(0.95)`) as cut-off. So we reject the null hypothesis if our test statistic is larger than 1.64.
4. The test statistic Z is $\sqrt{n} \times \bar{X}$, and its distribution is $N(\sqrt{n} \times \Delta, 1)$. We reject the null hypothesis if $Z >$ 1.64. Therefore, the power is

$$Pr(Z > 1.64) = Pr\left\{(\bar{X} - \sqrt{n} \times \Delta) > 1.64 - \sqrt{n} \times \Delta\right\} = Pr\left\{\tilde{Z} > 1.64 - \Delta\sqrt{n})\right\}$$

with $\tilde{Z} \sim N(0,1)$ under H_a. We want the power to be 80%. When taking a draw from a $N(0,1)$, we find that the chance of getting a value larger than -0.84 is equal to 80% (`qnorm(0.2)` or `qnorm(0.8,lower=FALSE)`). Therefore, to have 80% power to detect a mean of $\Delta = 0.5$, it follows from Pr($Z > -0.84$) = 0.80 that

$$1.64 - 0.5\sqrt{n} = -0.84 \Longleftrightarrow n = (2 \times (1.64 + 0.84))^2 = 24.6$$

Hence we need at least 25 samples. If you type

```
> power.t.test(delta=0.5,power=0.8,sd=1,alternative="one.sided",
               type="one.sample")
```

in R, the answer is 26.2. i. e. that 27 samples are needed. This is because R assumes that $\sigma = 1$ is the true but *unknown* standard deviation, and thus, has to be estimated from the data. In other words, it uses a t-distribution for the sample size calculations, instead of the normal distribution, and that's why the sample size needed is slightly larger. The price for not knowing the standard deviation is 2 extra samples!